

# Strong shock in the uniformly expanding medium

G.S. Bisnovatyi-Kogan \*

## Abstract

Propagation of the strong shock in the expanding Friedman universe is investigated using methods of dimension and similarity. Exact analytic solution of self-similar equations is obtained, determining dependences of the radius and velocity of the shock wave on time and radius. It is obtained, that in the expanding medium the velocity of shock decreases as  $\sim t^{-1/5}$ , what is slower than the shock velocity in the static uniform medium  $\sim t^{-3/5}$ . The radius of the shock wave in the expanding self-gravitating medium increases  $\sim t^{4/5}$ , more rapidly than the shock wave radius in the uniform non-gravitating medium  $\sim t^{2/5}$ . So, the shock propagates in the direction of decreasing density with larger speed, that in the static medium, due to accelerating action of the decreasing density, even in the presence of a self-gravitation.

*Keywords:* cosmology, strong shock wave, self-similar solution

## 1 Introduction

At early stages of star and galaxy formation we may expect strong explosions at last stages of evolution of very massive primordial stars, which enrich the matter with heavy elements. Detection of heavy elements at red shifts up to  $z \sim 10$  from

---

\*Space Research Institute, Russian Academy of Sciences, Moscow, Russia; National Research Nuclear University MEPhI, Moscow, Russia; and ICRANet, CBPF, Rio de Janeiro, Brazil.

GRB observations (GRB090423 at  $z \approx 8.2$ , GRB120923A at  $z \approx 8.5$ , GRB090429B with a photo- $z \approx 9.4$ ) [6] make plausible this suggestion. Propagation of a strong shock in the static uniform media was investigated by many authors [5], [7], but the finite analytic self-similar solution was obtained in [3]. Here we obtain a self-similar solution for the strong shock propagating through the uniform expanding media, corresponding to the Friedman solution for the flat universe [8].

## 2 Self-similar solution for a strong shock propagation in a uniform static medium

Let us first describe the self-similar solution of Sedov [3], following [1]. Hydrodynamic equations in spherically symmetric isentropic case are written as

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r}, \quad \frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial r} + \frac{2\rho v}{r} = 0, \quad \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial r} \right) \ln \frac{p}{\rho^\gamma} = 0. \quad (1)$$

Here  $v$  is a gas velocity in the laboratory coordinate system,  $\rho$  is the gas density,  $p$  is the gas pressure. Defining the variables with subscripts "1" and "2" corresponding to the values in front and behind the shock wave, we may write the relations on the shock, moving through the static media ( $v_1 = 0$ ) in the form [4]

$$\begin{aligned} v_2 &= \frac{2}{\gamma + 1} u_1 \left( 1 - \frac{c_1^2}{u_1^2} \right), \\ \rho_2 &= \frac{\gamma + 1}{\gamma - 1} \rho_1 \left( 1 + \frac{2}{\gamma - 1} \frac{c_1^2}{u_1^2} \right)^{-1}, \\ p_2 &= \frac{2}{\gamma + 1} \rho_1 u_1^2 \left( 1 - \frac{\gamma - 1}{2\gamma} \frac{c_1^2}{u_1^2} \right). \end{aligned} \quad (2)$$

Here  $u_1$  is the shock velocity relative to the unmoving gas in front ( $v_1 = 0$ ),  $\gamma$  is the adiabatic power,  $c_1$  is the sound velocity in the static gas,  $c_1^2 = \gamma \left( \frac{p_1}{\rho_1} \right)$ . In the strong shock the shock velocity is much larger than the sound velocity in the undisturbed gas,  $u_1 \gg c_1$ , and  $p_2 \gg \frac{\gamma+1}{\gamma-1} p_1$ , so we have from (2)

$$v_2 = \frac{2}{\gamma + 1}u_1, \quad \rho_2 = \frac{\gamma + 1}{\gamma - 1}\rho_1, \quad p_2 = \frac{2}{\gamma + 1}\rho_1 u_1^2, \quad c_2^2 = \frac{2\gamma(\gamma - 1)}{(\gamma + 1)^2}u_1^2. \quad (3)$$

There are only two parameters in the case of the strong shock: the density of the unperturbed gas  $\rho_1$ . and the total energy of the explosion  $E$ . From these parameters, together with the independent variables  $t, r$ , it is possible to construct only one non-dimensional combination  $r(\rho_1/Et^2)^{1/5}$ , what means that the problem has a self-similar solution. A position of the shock in the self-similar solution should correspond to fixed value of the self-similar coordinate, so that the distance of the shock to the center  $R$  may be written as

$$R = \beta \left( \frac{Et^2}{\rho_1} \right)^{1/5}, \quad (4)$$

where  $\beta$  is a number depending only on the adiabatic power  $\gamma$ . The velocity of the shock  $u_1$  in the laboratory frame, relative to the unperturbed gas is defined as

$$u_1 = \frac{dR}{dt} = \frac{2R}{5t} = \frac{2\beta E^{1/5}}{5\rho_1^{1/5}t^{3/5}}. \quad (5)$$

So, the velocity of the shock  $u_1$ , the velocity of the matter behind the shock  $v_2$ , moving through the constant density medium are decreasing with time  $\sim t^{-3/5}$ , and pressure behind the shock  $p_2 \sim t^{-6/5}$ .

Let us use a sound velocity variable  $c^2 = \gamma p/\rho$  instead of a pressure  $p$ . To find a full solution of the problem introduce non-dimensional variables behind the shock  $(V, G, Z)$  as

$$v = \frac{2r}{5t}V, \quad \rho = \rho_1 G, \quad c^2 = \frac{4r^2}{25t^2}Z. \quad (6)$$

depending on the self-similar variable  $\xi$ , defined as

$$\xi = \frac{r}{R(t)} = \frac{r}{\beta} \left( \frac{\rho_1}{Et^2} \right)^{1/5}. \quad (7)$$

The conditions (3) on the strong shock at  $r = R$ ,  $\xi = 1$ , in non-dimensional variables (6) are written as

$$V(1) = \frac{2}{\gamma + 1}, \quad G(1) = \frac{\gamma + 1}{\gamma - 1}, \quad Z(1) = \frac{2\gamma(\gamma - 1)}{(\gamma + 1)^2}. \quad (8)$$

In non-dimensional variables (6) the original system (1) is written as

$$Z \left( \frac{d \ln Z}{d \ln \xi} + \frac{d \ln G}{d \ln \xi} + 2 \right) + \gamma(V - 1) \frac{dV}{d \ln \xi} = \gamma V \left( \frac{5}{2} - V \right), \quad (9)$$

$$\frac{dV}{d \ln \xi} - (1 - V) \frac{d \ln G}{d \ln \xi} = -3V, \quad (10)$$

$$\frac{d \ln Z}{d \ln \xi} - (\gamma - 1) \frac{d \ln G}{d \ln \xi} = -\frac{5 - 2V}{1 - V}. \quad (11)$$

Here we used relations

$$\frac{\partial \xi}{\partial t} \Big|_r = -\frac{2\xi}{5t}, \quad \frac{\partial \xi}{\partial r} \Big|_t = \frac{\xi}{r}. \quad (12)$$

## 2.1 Energy conservation integral

The full analytic solution of the problem was obtained by L.I.Sedov, who found the expression for the full energy integral  $E$ , what permitted to omit the equation (9), see [3]. The conservation of full energy in this problem is a result of the propagation through the medium with zero pressure, what means zero external work. Due to self-similarity there should be conservation of the total energy inside any sphere of smaller radius, which increases with time according to the law  $\xi = \text{const}$ , with any value of the *const*. Points on the sphere of the radius  $r$  expanding according this law, have a velocity  $v_n = \frac{2r}{5t} = u_1 \frac{r}{R}$ . A simple physical derivation of the energy conservation law was given in the book [1].

The change of the energy of the spherical region with a radius  $r$ , during the time  $dt$ , is determined by its loss due to the flux of the energy  $dq_-$  carried away

by the matter, expanding with the velocity  $v$ . Using Bernoulli integral, the flux through the spherical surface with radius  $r$  during the time  $dt$  is written as

$$dq_- = dt 4\pi r^2 \rho v (w + \frac{v^2}{2}) \quad (13)$$

The gain of the energy  $dq_+$  is due to increase of the volume of this sphere during the time  $dt$  by the value  $(dt v_n 4\pi r^2)$ , containing the energy

$$dq_+ = dt 4\pi r^2 \rho v_n (\varepsilon + \frac{v^2}{2}), \quad w = \varepsilon + \frac{p}{\rho}. \quad (14)$$

From the equality  $q_- = q_+$  we have

$$v(w + \frac{v^2}{2}) = v_n(\varepsilon + \frac{v^2}{2}). \quad (15)$$

From thermodynamic relations we have [2]

$$\varepsilon = \frac{c^2}{\gamma(\gamma - 1)}, \quad w = \frac{c^2}{\gamma - 1}, \quad (16)$$

so that (15) is written as

$$v \left( \frac{c^2}{\gamma - 1} + \frac{v^2}{2} \right) = v_n \left( \frac{c^2}{\gamma(\gamma - 1)} + \frac{v^2}{2} \right). \quad (17)$$

It follows from (5),(6) the relation  $\frac{v}{v_n} = V$ , and we have from (17)

$$\frac{c^2}{\gamma - 1} (V - \frac{1}{\gamma}) = \frac{v^2}{2} (1 - V) \quad (18)$$

Using (6) we write the energy conservation law in the form [4],[1]

$$Z = \frac{\gamma(\gamma - 1)(1 - V)}{2(\gamma V - 1)} V^2. \quad (19)$$

Evidently, that at the shock  $r = R$ ,  $\xi = 1$ , with  $V(1)$  and  $Z(1)$  from (8), the energy integral (19) becomes an identity. At existence of the energy integral (19) we may use only two differential equations (10) and (11) for finding a solution

of the problem. It was shown in [3], that this solution may be obtained in the analytical form.

## 2.2 Exact solution

Following [3],[4], we look for an analytic solution of (10),(11), where the variable  $Z$  is determined by the algebraic relation (19). Excluding  $\frac{d \ln G}{d \ln \xi}$  from (10),(11), we obtain the equation

$$(\gamma - 1) \frac{dV}{d \ln \xi} - (1 - V) \frac{d \ln Z}{d \ln \xi} = 5 - 3\gamma V + V. \quad (20)$$

From (19) we find

$$\frac{d \ln Z}{d \ln \xi} = -\frac{1}{1 - V} \frac{dV}{d \ln \xi} + 2 \frac{1}{V} \frac{dV}{d \ln \xi} - \frac{\gamma}{\gamma V - 1} \frac{dV}{d \ln \xi}. \quad (21)$$

After substituting in (20) we find the following equation for the variable  $V$

$$\left( \gamma + 1 - \frac{2}{V} + \frac{\gamma - 1}{\gamma V - 1} \right) \frac{dV}{d \ln \xi} = 5 - 3\gamma V + V. \quad (22)$$

Using relations

$$\begin{aligned} -\frac{2}{V(5 - 3\gamma V + V)} &= -\frac{2}{5V} + \frac{2}{5} \frac{1 - 3\gamma}{5 - 3\gamma V + V}, \\ \frac{\gamma - 1}{(\gamma V - 1)(5 - 3\gamma V + V)} &= \frac{\gamma - 1}{2\gamma + 1} \frac{\gamma}{\gamma V - 1} + \frac{\gamma - 1}{2\gamma + 1} \frac{3\gamma - 1}{5 - 3\gamma V + V}, \end{aligned} \quad (23)$$

we obtain a solution of (22) in the form

$$\begin{aligned} &\frac{\gamma + 1}{1 - 3\gamma} \ln(5 - 3\gamma V + V) - \frac{2}{5} \ln V + \frac{2}{5} \ln(5 - 3\gamma V + V) \\ &+ \frac{\gamma - 1}{2\gamma + 1} \ln(\gamma V - 1) - \frac{\gamma - 1}{2\gamma + 1} \ln(5 - 3\gamma V + V) = \ln \xi + \text{Const}. \end{aligned} \quad (24)$$

Using the boundary condition for  $\xi = 1$  from (8), we obtain the solution for  $V(\xi)$  as [1]

$$\left(\frac{\gamma+1}{2}V\right)^{-2} \left[\frac{\gamma+1}{\gamma-1}(\gamma V-1)\right]^{\nu_1} \left[\frac{\gamma+1}{7-\gamma}(5-3\gamma V+V)\right]^{\nu_2} = \xi^5, \quad (25)$$

with

$$\nu_1 = 5\frac{\gamma-1}{2\gamma+1}, \quad \nu_2 = -\frac{13\gamma^2-7\gamma+12}{(3\gamma-1)(2\gamma+1)}. \quad (26)$$

For finding a solution for  $G(\xi)$  we write the equations (10) and (22) in the form

$$1 - (1-V)\frac{d \ln G}{dV} = -3V\frac{d \ln \xi}{dV}, \quad (27)$$

$$\frac{d \ln \xi}{dV} = \frac{\gamma+1-\frac{2}{V}+\frac{\gamma-1}{\gamma V-1}}{5-3\gamma V+V}. \quad (28)$$

The equation for  $G(V)$  is obtained from these equations in the form

$$1 - (1-V)\frac{d \ln G}{dV} = -3V\frac{\gamma+1-\frac{2}{V}+\frac{\gamma-1}{\gamma V-1}}{5-3\gamma V+V}, \quad (29)$$

which reduces to

$$(1-V)\frac{d \ln G}{dV} = \frac{2(1+2V)+3\frac{1-V}{\gamma V-1}}{5-3\gamma V+V}. \quad (30)$$

Using relations

$$\begin{aligned} \frac{2+4V}{(5-3\gamma V+V)(1-V)} &= \frac{2}{2-\gamma} \frac{1}{1-V} - 2\frac{\gamma+3}{2-\gamma} \frac{1}{5-3\gamma V+V}, \\ \frac{3}{(\gamma V-1)(5-3\gamma V+V)} &= \frac{3\gamma}{2\gamma+1} \frac{1}{\gamma V-1} + 3\frac{3\gamma-1}{2\gamma+1} \frac{1}{5-3\gamma V+V}, \end{aligned} \quad (31)$$

we obtain the solution of (30) in the form

$$\begin{aligned} \ln G &= -\frac{2}{2-\gamma} \ln(1-V) + \frac{3}{2\gamma+1} \ln(\gamma V-1) \\ &+ \frac{13\gamma^2-7\gamma+12}{(2-\gamma)(3\gamma-1)(2\gamma+1)} \ln(5-3\gamma V+V) + \mathbf{const}. \end{aligned} \quad (32)$$

The **const** is found from the boundary conditions (8), and finally we obtain the solution for  $G(V)$  in the form

$$G(V) = \frac{\gamma+1}{\gamma-1} \left[ \frac{\gamma+1}{\gamma-1} (1-V) \right]^{\sigma_1} \left[ \frac{\gamma+1}{\gamma-1} (\gamma V - 1) \right]^{\sigma_2} \times \left[ \frac{\gamma+1}{7-\gamma} (5 - 3\gamma V + V) \right]^{\sigma_3}, \quad (33)$$

where

$$\sigma_1 = -\frac{2}{2-\gamma}, \quad \sigma_2 = \frac{3}{2\gamma+1}, \quad \sigma_3 = \frac{13\gamma^2 - 7\gamma + 12}{(2-\gamma)(3\gamma-1)(2\gamma+1)}. \quad (34)$$

The function  $Z(V)$  is determined by the integral (19).

The constant  $\beta$  in the definition of the non-dimensional radius  $\xi$  in (7) is obtained from the energy integral  $E$  for the known energy of the explosion. The conserving value of the energy behind the shock is determined as [1]

$$E = \int_0^{R(t)} \rho \left[ \frac{v^2}{2} + \frac{c^2}{\gamma(\gamma-1)} \right] 4\pi r^2 dr. \quad (35)$$

In non-dimensional variables (6) this relation reduces to the equation for the constant  $\beta$

$$1 = \beta^5 \frac{16\pi}{25} \int_0^1 G \left[ \frac{V^2}{2} + \frac{Z}{\gamma(\gamma-1)} \right] \xi^4 d\xi \quad (36)$$

The value of  $\beta$ , following from this relation is of the order of unity [1], for  $\gamma = \frac{7}{5}$  we have  $\beta = 1.033$ .

### 3 Strong shock in a uniform expanding medium

Let us consider a uniformly expanding medium, corresponding to the flat expanding Friedman universe [8], with a density  $\rho_1$ , and expansion velocity  $v_1$  as

$$\rho_1 = \delta/t^2, \quad v_1 = 2r/3t. \quad (37)$$



This is an exact solution for a uniform expanding self-gravitating medium with velocity tending to zero at time infinity. The equations in spherical coordinates in this case are written as

$$\begin{aligned} \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} - \frac{G_g m}{r^2}, \quad \frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial r} + \frac{2\rho v}{r} = 0, \\ \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial r} \right) \ln \frac{p}{\rho^\gamma} &= 0, \quad m = \frac{4\pi}{3} \rho r^3. \end{aligned} \quad (38)$$

The solution (37) satisfies equations (38) at ( $G_g$  - gravitational constant)

$$\delta = \frac{1}{6\pi G_g}, \quad \rho_1 = \frac{1}{6\pi G_g t^2}, \quad \frac{G_g m}{r^2} = \frac{2}{9} \frac{r}{t^2}. \quad (39)$$

The newtonian solution for the flat expanding universe is valid physically in the region where  $v_1 \ll c_{\text{light}}$ ,  $c \ll c_{\text{light}}$ . For the case of a point explosion with the energy  $E$ , the number of parameters is the same as in the previous static medium ( $\delta$ ,  $E$ ), therefore we look for a self-similar solution for the case of a strong shock motion. The non-dimensional combination in the case of a uniformly expanding medium is written as  $r(\delta/Et^4)^{1/5}$ . A position of the shock in the self-similar solution should correspond to the fixed value of the self-similar coordinate, so that the distance of the shock to the center  $R$  may be written as

$$R = \beta \left( \frac{Et^4}{\delta} \right)^{1/5}, \quad (40)$$

where  $\beta$  is a number depending only on the adiabatic power  $\gamma$ . The velocity of the shock  $u$  in the laboratory frame, for the unperturbed expanding gas is defined as

$$u = \frac{dR}{dt} = \frac{4R}{5t} = \frac{4\beta E^{1/5}}{5\delta^{1/5} t^{1/5}}. \quad (41)$$

The velocity of the shock  $u$ , the velocity of the matter behind the shock  $v_2$ , moving through the uniformly expanding medium (37), are decreasing with time  $\sim t^{-1/5}$ , and the pressure behind the shock  $p_2$  is decreasing  $\sim t^{-2/5}$ , what is considerably slower than in the case of the constant density medium. It happens due to the

fact, that the background density is decreasing with time, and the resistance to the shock propagation is decreasing also.

The Euler equations of motion (1) are valid in this case, but the conditions on the strong shock discontinuity should be written, with account of the expansion in the laboratory frame as

$$\begin{aligned} v_2 &= \frac{2}{\gamma+1}u + \frac{\gamma-1}{\gamma+1}v_1^{sh}, \quad \rho_2 = \frac{\gamma+1}{\gamma-1}\rho_1, \\ p_2 &= \frac{2}{\gamma+1}\rho_1(u - v_1^{sh})^2, \quad c_2^2 = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}(u - v_1^{sh})^2. \end{aligned} \quad (42)$$

Here  $v_1^{sh} = \frac{2R}{3t}$  is the unperturbed expansion velocity on the shock level. We introduce non-dimensional variables behind the shock as

$$v = \frac{4r}{5t}V, \quad \rho = \frac{\delta}{t^2}G, \quad c^2 = \frac{16r^2}{25t^2}Z, \quad (43)$$

depending on the self-similar variable  $\xi$ , defined as

$$\xi = \frac{r}{R(t)} = \frac{r}{\beta} \left( \frac{\delta}{Et^4} \right)^{1/5}. \quad (44)$$

The conditions (42) on the strong shock at  $r = R$ ,  $\xi = 1$ , in non-dimensional variables (43) are written as

$$V(1) = \frac{5\gamma+7}{6(\gamma+1)}, \quad G(1) = \frac{\gamma+1}{\gamma-1}, \quad Z(1) = \frac{\gamma(\gamma-1)}{18(\gamma+1)^2}. \quad (45)$$

In non-dimensional variables (43) the original system (1) is written as

$$Z \left( \frac{d \ln Z}{d \ln \xi} + \frac{d \ln G}{d \ln \xi} + 2 \right) + \gamma(V-1) \frac{dV}{d \ln \xi} = \gamma V \left( \frac{5}{4} - V \right) - \frac{25}{72} \gamma G, \quad (46)$$

$$\frac{dV}{d \ln \xi} - (1-V) \frac{d \ln G}{d \ln \xi} = -3V + \frac{5}{2}, \quad (47)$$

$$\frac{d \ln Z}{d \ln \xi} - (\gamma-1) \frac{d \ln G}{d \ln \xi} = -\frac{5-2V-\frac{5}{2}\gamma}{1-V}. \quad (48)$$

Here we used relations

$$\frac{\partial \xi}{\partial t}|_r = -\frac{4\xi}{5t}, \quad \frac{\partial \xi}{\partial r}|_t = \frac{\xi}{r}. \quad (49)$$

### 3.1 Construction of the first integral of the problem

The system of three equations (46)-(48), describing the self-similar solution of the problem, has a first integral, similar to the energy conservation integral in the case of the static background medium (19). To find this integral, we consider a situation in the coordinate system, where the background medium is locally static on the shock at  $r = R(t)$ . In this non-inertial coordinate system it is possible to construct the combination of functions, representing the first integral. We cannot call this integral as an energy integral, because the the system at which it has the same form as (19) is not inertial. Due to self-similarity there should be conservation of this integral inside any sphere of smaller radius  $r \leq R(t)$ . The frame which is locally comoving at  $r = R$ , has a velocity  $v_1^{sh} = \frac{2R}{3t}$  relative to the laboratory frame, and at  $r \leq R(t)$  we should use instead, due to self-similarity, the velocity  $v_1^{sh} \frac{r}{R} = \frac{2r}{3t}$ , like in the case of the static background. This velocity should be subtracted from all velocities in the expression for the first integral, which has the form (17). At  $r \leq R(t)$  the velocity  $v_n = u \frac{r}{R} = \frac{4r}{5t}$ . We have than

$$(v - v_1^{sh} \frac{r}{R}) \left( \frac{c^2}{\gamma - 1} + \frac{(v - v_1^{sh} \frac{r}{R})^2}{2} \right) = (v_n - v_1^{sh} \frac{r}{R}) \left( \frac{c^2}{\gamma(\gamma - 1)} + \frac{(v - v_1^{sh} \frac{r}{R})^2}{2} \right). \quad (50)$$

In the non-dimensional variables (43) we have from (50)

$$\begin{aligned} & \left( \frac{4r}{5t} V - \frac{2r}{3t} \right) \left[ \frac{Z}{\gamma - 1} \frac{16r^2}{25t^2} + \frac{1}{2} \left( \frac{4r}{5t} V - \frac{2r}{3t} \right)^2 \right] = \\ & \left( \frac{4r}{5t} - \frac{2r}{3t} \right) + \left[ \frac{Z}{\gamma(\gamma - 1)} \frac{16r^2}{25t^2} + \frac{1}{2} \left( \frac{4r}{5t} V - \frac{2r}{3t} \right)^2 \right]. \end{aligned} \quad (51)$$

This relation reduces to

$$(V - \frac{5}{6}) \left[ \frac{Z}{\gamma - 1} + (V - \frac{5}{6})^2 \right] = (1 - \frac{5}{6}) \left[ \frac{Z}{\gamma(\gamma - 1)} + (V - \frac{5}{6})^2 \right], \quad (52)$$

from what follows the first integral in the form

$$Z = \frac{(\gamma - 1)(1 - V)(V - \frac{5}{6})^2}{2(V - \frac{5}{6} - \frac{1}{6\gamma})}. \quad (53)$$

At the shock  $r = R$ ,  $\xi = 1$ , with  $Z(1)$  and  $V(1)$  from (26), the first integral becomes an identity, what confirms its correctness. At existence of the first integral (53) we may use only two differential equations (47) and (48) for finding a solution of the problem, like in the classical Sedov case.

### 3.2 Exact solution for the expanding meduim

Similar to the static background medium in the previous section (see [3],[4]), we look for an analytic solution of (47),(48), where the variable  $Z$  is determined by the algebraic relation (53). Excluding  $\frac{d \ln G}{d \ln \xi}$  from (47),(48), we obtain the equation

$$(\gamma - 1) \frac{dV}{d \ln \xi} - (1 - V) \frac{d \ln Z}{d \ln \xi} = \frac{5}{2} - 3\gamma V + V. \quad (54)$$

From (53) we find

$$\frac{d \ln Z}{d \ln \xi} = -\frac{1}{1 - V} \frac{dV}{d \ln \xi} + \frac{2}{V - \frac{5}{6}} \frac{dV}{d \ln \xi} - \frac{\gamma}{\gamma V - \frac{5\gamma}{6} - \frac{1}{6}} \frac{dV}{d \ln \xi}. \quad (55)$$

After substituting in (54) we find the following equation for the variable  $V$

$$\left( \gamma + 1 - \frac{2}{6V - 5} + \frac{\gamma - 1}{6\gamma V - 5\gamma - 1} \right) \frac{dV}{d \ln \xi} = \frac{5}{2} - 3\gamma V + V. \quad (56)$$

Using relations

$$\begin{aligned} \frac{2}{(6V - 5)(\frac{5}{2} - 3\gamma V + V)} &= \frac{12}{(20 - 15\gamma)(6V - 5)} \\ &+ \frac{2(3\gamma - 1)}{(20 - 15\gamma)(\frac{5}{2} - 3\gamma V + V)}, \end{aligned} \quad (57)$$

$$\begin{aligned} \frac{1}{(6\gamma V - 5\gamma - 1)(\frac{5}{2} - 3\gamma V + V)} &= \frac{1}{17\gamma - 15\gamma^2 + 1} \frac{6\gamma}{6\gamma V - 5\gamma - 1} \\ &+ \frac{1}{17\gamma - 15\gamma^2 + 1} \frac{3\gamma - 1}{\frac{5}{2} - 3\gamma V + V}, \end{aligned} \quad (58)$$

we obtain a solution of (56) in the form

$$\begin{aligned} & - \frac{2}{20 - 15\gamma} \ln(V - \frac{5}{6}) + \frac{\gamma - 1}{17\gamma - 15\gamma^2 + 1} \ln(V - \frac{5}{6} - \frac{1}{6\gamma}) \\ & + \left[ -\frac{\gamma + 1}{3\gamma - 1} - \frac{\gamma - 1}{17\gamma - 15\gamma^2 + 1} + \frac{2}{20 - 15\gamma} \right] \ln \left( V - \frac{5}{6\gamma - 2} \right) = \ln \xi + \text{const.} \end{aligned} \quad (59)$$

Using the boundary condition for  $\xi = 1$  from (45), we obtain the solution for  $V(\xi)$  as

$$\left[ (\gamma + 1)(3V - \frac{5}{2}) \right]^{\mu_1} \left[ \frac{\gamma + 1}{\gamma - 1} (6\gamma V - 5\gamma - 1) \right]^{\mu_2} \left[ 6(\gamma + 1) \frac{3\gamma V - V - \frac{5}{2}}{15\gamma^2 + \gamma - 22} \right]^{\mu_3} = \xi, \quad (60)$$

with

$$\begin{aligned} \mu_1 &= \frac{2}{15\gamma - 20}, \quad \mu_2 = \frac{\gamma - 1}{17\gamma - 15\gamma^2 + 1}, \\ \mu_3 &= -\frac{\gamma + 1}{3\gamma - 1} - \frac{\gamma - 1}{17\gamma - 15\gamma^2 + 1} + \frac{2}{20 - 15\gamma}. \end{aligned} \quad (61)$$

For finding a solution for  $G(\xi)$  we write the equations (47) and (56) in the form

$$1 - (1 - V) \frac{d \ln G}{dV} = -(3V - \frac{5}{2}) \frac{d \ln \xi}{dV}, \quad (62)$$

$$\frac{d \ln \xi}{dV} = \frac{\gamma + 1 - \frac{2}{6V - 5} + \frac{\gamma - 1}{6\gamma V - 5\gamma - 1}}{\frac{5}{2} - V(3\gamma - 1)}. \quad (63)$$

The equation for  $G(V)$  is obtained from these equations in the form

$$1 - (1 - V) \frac{d \ln G}{dV} = -(3V - \frac{5}{2}) \frac{\gamma + 1 - \frac{2}{6V - 5} + \frac{\gamma - 1}{6\gamma V - 5\gamma - 1}}{\frac{5}{2} - V(3\gamma - 1)}. \quad (64)$$

It is reduces to

$$(1-V)\frac{d \ln G}{dV} = -\frac{4}{3\gamma-1} + \frac{\frac{15\gamma}{2}-10}{3\gamma-1} \frac{\gamma+1}{(3\gamma-1)V-\frac{5}{2}} \\ + \frac{1}{(3\gamma-1)V-\frac{5}{2}} - \frac{\gamma-1}{6\gamma V-5\gamma-1} \frac{3V-\frac{5}{2}}{V(3\gamma-1)-\frac{5}{2}}. \quad (65)$$

Here the relation was used:

$$\frac{3V-\frac{5}{2}}{\frac{5}{2}-3\gamma V+V} = \frac{1}{3\gamma-1} \left( -3 + \frac{20-15\gamma}{2(\frac{5}{2}-3\gamma V+V)} \right). \quad (66)$$

Using algebraic relations (58),(66),

$$\frac{1}{(1-V)(6\gamma V-5\gamma-1)} = \frac{1}{\gamma-1} \left( \frac{1}{1-V} + \frac{6\gamma}{6\gamma V-5\gamma-1} \right), \quad (67)$$

$$\frac{1}{(1-V)(\frac{5}{2}-3\gamma V+V)} = \frac{1}{3\gamma-\frac{7}{2}} \left( -\frac{1}{1-V} + \frac{3\gamma-1}{\frac{5}{2}-3\gamma V+V} \right), \quad (68)$$

we obtain the solution of (65) in the form

$$\ln G = \kappa_1 \ln(1-V) + \kappa_2 \ln(V - \frac{5\gamma+1}{6\gamma}) + \kappa_3 \ln(V - \frac{5}{6\gamma-2}) + \mathbf{const}_1. \quad (69)$$

Here

$$\kappa_1 = \frac{7}{3\gamma-1} - \frac{2}{6\gamma-7} + \frac{(15\gamma-20)(\gamma-1)}{(6\gamma-7)(15\gamma^2-17\gamma-1)} \\ - \frac{3\gamma(15\gamma-20)}{(3\gamma-1)(15\gamma^2-17\gamma-1)} - \frac{15\gamma-20}{3\gamma-1} \frac{\gamma+1}{6\gamma-7}, \quad (70) \\ \kappa_2 = -\frac{3}{3\gamma-1} + \frac{3\gamma(15\gamma-20)}{(3\gamma-1)(15\gamma^2-17\gamma-1)}. \\ \kappa_3 = \frac{2}{6\gamma-7} - \frac{(15\gamma-20)(\gamma-1)}{(6\gamma-7)(15\gamma^2-17\gamma-1)} + \frac{15\gamma-20}{3\gamma-1} \frac{\gamma+1}{6\gamma-7},$$

The  $\mathbf{const}_1$  is found from the boundary conditions (45), and finally we obtain the solution for  $G(V)$  in the form

$$G(V) = \frac{\gamma+1}{\gamma-1} \left[ 6 \frac{(\gamma+1)(1-V)}{\gamma-1} \right]^{\kappa_1} \left[ \frac{\gamma+1}{\gamma-1} (6\gamma V-5\gamma-1) \right]^{\kappa_2} \\ \times \left( \frac{3(\gamma+1)}{15\gamma^2+\gamma-22} [(6\gamma-2)V-5] \right)^{\kappa_3}. \quad (71)$$

The function  $Z(V)$  is determined by the integral (53).

## 4 Discussion

The constant  $\beta$  in the definition of the non-dimensional radius  $\xi$  in (44) is obtained from the explosion energy integral  $E$ . The conserving value of the explosion energy behind the shock in the uniformly expanding medium, with velocity and density distributions (37), is determined as

$$E = \int_0^{R(t)} \rho \left[ \frac{\left(v - \frac{2r}{3t}\right)^2}{2} + \frac{c^2}{\gamma(\gamma - 1)} \right] 4\pi r^2 dr. \quad (72)$$

In non-dimensional variables (43) this relation reduces to the equation for the constant  $\beta$

$$1 = \beta^5 \frac{64\pi}{25} \int_0^1 G \left[ \frac{\left(V - \frac{5}{6}\right)^2}{2} + \frac{Z}{\gamma(\gamma - 1)} \right] \xi^4 d\xi \quad (73)$$

The gravitational energy of the self-gravitation is not included directly in the explosion energy, but implicitly it is present in the decreasing expansion velocity of the background solution (39).

It follows from the self-similar solution, that in the expanding medium the velocity of shock from (41) decreases as  $\sim t^{-1/5}$ , what is much slower than the shock velocity in the static uniform medium  $\sim t^{-3/5}$ , according to Sedov solution (5). Correspondingly the radius of the shock wave in the expanding self-gravitating medium increases  $\sim t^{4/5}$ , more rapidly than the shock wave radius in the uniform non-gravitating medium  $\sim t^{2/5}$ . It means, that the shock propagates in the direction of decreasing density with larger speed, than in the static medium, due to accelerating action of the decreasing density, even in the presence of a self-gravitation.

## Acknowledgments

This work was partially supported by the Russian Federation President Grant for Support of Leading Scientific Schools, Grant No. NSh-261.2014.2, RFBR grant 14-02-00728, RAN Program Formation and evolution of stars and galaxies, and CAPES-ICRANet Programm. Author is grateful to A.D. Chernin for stimulating discussions, and Ivan Siutsou for help.

## References

- [1] Landau L. D., Lifshitz E. M. (1988) *Hydrodynamics*. Nauka, Moscow (in Russian)
- [2] Landau L. D., Lifshitz E. M. (1964) *Statistical Physics*. Nauka, Moscow (in Russian).
- [3] Sedov L.I. (1946) Doklady Acad. USSR **52**, No.1.
- [4] Sedov L.I. (1977) *Metody podobiya i razmernostei v mekhanike*. Nauka, Moscow (in Russian).
- [5] Stanyukovich K.P. (1955) *Nonstationary motion of continuous media*. Gostekhizdat. Moscow (in Russian).
- [6] Tanvir N. (2013) arXiv:1307.6156v1.
- [7] Taylor G.I. (1950), Proc. Roy. Soc. **A201**, 175.
- [8] Zeldovich Ya. B., Novikov I.D. (1983) *Relativistic astrophysics. Volume 2. The structure and evolution of the universe*. Chicago, IL, University of Chicago Press.